Malaysian Journal of Mathematical Sciences 19(4): 1553–1565 (2025)
https://doi.org/10.47836/mjms.19.4.19

Malaysian Journal of Mathematical Sciences

Journal homepage: https://mjms.upm.edu.my

Identification Number of Antiprism Graphs

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Received: 15 January 2025
Accepted: 3 July 2025

Abstract

Let G be a simple, connected, undirected, nontrivial graph and let c be a coloring of the vertices of G using the colors red and white such that not all the vertices are colored white. For each vertex v of G, we can assign a vector code $\vec{d}(v) = (a_1, a_2, \dots, a_{\text{diam}(G)})$, where diam$(G)$ is the diameter of G and, for each $i = 1, 2, \dots, \text{diam}(G)$, the component a_i is equal to the number of red vertices whose distance to v is i. Then, c is said to be an ID-coloring of G if $\vec{d}(v) \neq \vec{d}(w)$ for any pair of distinct vertices v and w. If G has an identification coloring, it is called an ID-graph and its identification number ID(G) is defined to be the least number of red vertices needed to construct an ID-coloring of G. An ID-coloring of G can be used to distinguish its vertices from each other and this is particularly interesting when G has nontrivial automorphisms. Indeed, identification colorings have been studied for paths, cycles, grids, prisms, and some tree families, including caterpillar and lobster graphs. In this paper, we consider identification colorings for antiprism graphs, which have not been previously studied in relation to this topic. An antiprism graph is a graph that corresponds to the skeleton of an antiprism. Antiprism graphs are known to be vertex-transitive; thus, they possess nontrivial automorphisms. Our results include a characterization of all ID-antiprisms and the determination of the identification number of any ID-antiprism.

Keywords: identification coloring; multiset resolving set; antiprism graphs.

1 Introduction

Consider a simple, connected, nontrivial graph G. A red-white coloring of G is a vertex coloring of G where each vertex is colored red or white, and not all vertices have been colored white. Such a coloring induces a vector code $\vec{d}(v)$ for each vertex v, where $\vec{d}(v) = (a_i)_{i\in\{1,\ldots,\text{diam}(G)\}}$ and each a_i is equal to the number of red vertices whose distance to v is i. Chartrand et al. [1] have defined an identification coloring (or ID-coloring) of G to be a red-white coloring of G in which each vertex does not have the same vector code as other vertices. If G has at least one ID-coloring, then it is called an ID-graph and its identification number (or ID number), denoted by $\text{ID}(G)$, is the least number of red vertices needed to construct an ID-coloring of G.

In Hakanen dan Yero [4], the notion of ID-colorings has been shown to be equivalent to the notion of multiset resolving sets. This was first introduced by Saenpholphat in [8]. Independently, the same notion was reintroduced later by Simanjuntak et al. [10]. More precisely, we can also define the multiset code $M(v)$ to be the multiset containing the distances of v to all the red vertices in G. Then, in an ID-coloring, each vertex also does not have the same multiset code as other vertices; and the converse also holds. In Figure 1 and Table 1, we present an example of an ID-coloring and the vector codes and multiset codes of the vertices respectively.

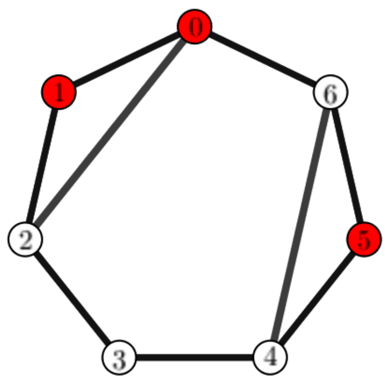

Figure 1: An ID-coloring of a graph.

Table 1: The vector codes and multiset codes of the vertices of an ID-coloring of a graph.

v	$\vec{d}(v)$	$M(v)$
0	$(1,1,0)$	$\{0,1,2\}$
1	$(1,0,1)$	$\{0,1,3\}$
2	$(2,0,1)$	$\{1,1,3\}$
3	$(0,3,0)$	$\{2,2,2\}$
4	$(1,1,1)$	$\{1,2,3\}$
5	$(0,1,1)$	$\{0,2,3\}$
6	$(2,1,0)$	$\{1,1,2\}$

Chartrand et al. [1] have introduced the notion of ID-colorings as an approach for solving the problem of uniquely identifying the vertices of a graph; that is, if G has an ID-coloring, then the vertices of G can be distinguished using their vector codes (or equivalently, their multiset codes).

The vertex identification problem is particularly interesting when the graph has nontrivial automorphisms. Indeed, previous studies on this topic have focused on graph families with nontrivial automorphisms. For example, ID-colorings of paths and cycles were investigated in [1]. On the other hand, Kono and Zhang [5] focused on ID-colorings of grids and prisms. Moreover, Marcelo et al. [6] studied the identification spectra (a related notion to ID-colorings) of prisms.

In this paper, we continue the theme of previous studies in this area by considering ID-colorings of antiprism graphs, which are known to have non-trivial automorphisms. Our objectives are to characterize all antiprism graphs that are ID-graphs and to determine the identification number of such antiprism graphs. In Section 2, we cover some preliminaries, including the definition and notations for antiprism graphs as well as some properties necessary for later discussions. In Section 3, we present our main result and its proof, which is split into several lemmas that are detailed in the succeeding subsections.

2 Preliminaries

We first introduce some notations. Suppose we have a graph G and a red-white coloring of G in which at least two vertices have been colored red. We define $\min M(v), \max M(v), \text{sum}(v)$, and $\text{sum}_2(v)$ to be the minimum element, the maximum element, the sum of all elements, and the sum of the two least elements, respectively, of the multiset code $M(v)$ of the vertex v. It is easy to observe that for two distinct vertices v and w, if $\min M(v) \neq \min M(w)$ or $\max M(v) \neq \max M(w)$ or $\text{sum}(v) \neq \text{sum}(w)$ or $\text{sum}_2(v) \neq \text{sum}_2(w)$, then $M(v) \neq M(w)$.

The following theorem compiles relevant results due to Chartrand et al., namely Proposition 2.2, Theorem 2.8, and Corollary 4.2 of [1].

Theorem 2.1.

(*a*) *There is no ID-coloring of a connected graph with exactly two red vertices.*

(*b*) *Any connected graph with at least two vertices has ID number is equal to 1 if and only if it is a path.*

(*c*) *Suppose G is vertex-transitive, connected, and $|V(G)| = n \geq 2$. If G has an ID-coloring in which exactly r vertices are colored red, then G also has an ID-coloring in which exactly $n - r$ vertices are colored red.*

An n-antiprism, $n \geq 3$, is a polyhedron that consists of two congruent n-sided polygons that are separated by a ring (or band) of $2n$ isosceles triangles [2, pg. 13]. The n-antiprism graph, which we denote by AP_n, is the graph that corresponds to the skeleton of the n-antiprism [7]. See Figure 2 for example.

Now, let $\mathbb{Z}_k$, $k \geq 3$, be the group of integers modulo k. Recall that the circulant graph $Ci_k(1, 2)$ has $V(Ci_k(1, 2)) = \mathbb{Z}_k$ and $E(Ci_k(1, 2)) = \{ij : i, j \in \mathbb{Z}_k \text{ and } i - j \in \{\pm 1, \pm 2\}\}$, where additive inverses are taken in $\mathbb{Z}_k$. For any integer $n \geq 3$, it can be easily observed that the antiprism graph $AP(n)$ is isomorphic to the circulant graph $Ci_{2n}(1, 2)$. Thus, for the rest of the paper, we use,

$$V(AP_n) = \mathbb{Z}_{2n} = \{0, 1, 2, \ldots, 2n-2, 2n-1\},$$

and

$$E(AP_n) = \{ij : i, j \in \mathbb{Z}_{2n} \text{ and } (i - j) \in \{\pm 1, \pm 2\}\}.$$

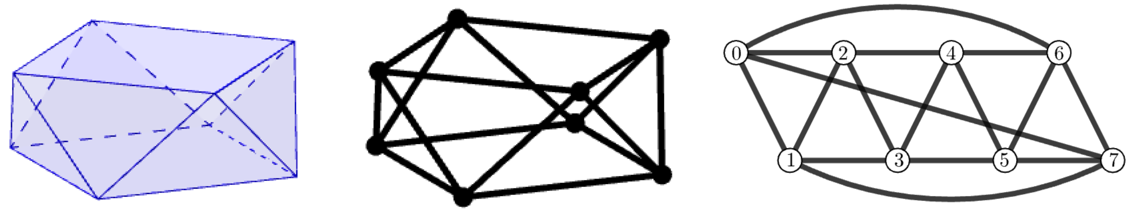

Figure 2: The 4-antiprism (left) and the 4-antiprism graph (middle), which is isomorphic to the circulant graph $Ci_8(1, 2)$ (right).

It is worth noting that circulant graphs have been studied with respect to different topics. For example, in El-Shanawany and El-Mesady [3], cyclic orthogonal double covers of circulant graphs were studied. On the other hand, Saleem et al. [9] studied antiprism graphs in relation to their metric dimension, which is closely related to the notion of multiset dimension,

Note that any antiprism graph is vertex-transitive. From the rightmost graph in Figure 2, it can be observed that an antiprism graph has a cycle induced by the even vertices and another cycle induced by the odd vertices; more specifically, these cycles are $0 \to 2 \to 4 \to 6 \to 0$ and $1 \to 3 \to 5 \to 7 \to 1$, respectively. Furthermore, the shortest path between vertices of the same parity is in one of these two cycles. From this, we can derive the following distance formula for vertices in an antiprism graph.

Observation 2.2. *Let i, j be vertices in an n-antiprism graph, where $n \geq 3$. Then,*

$$d(i,j) = \min\left\{ \left\lceil \frac{|j-i|}{2} \right\rceil, \left\lceil \frac{2n-|j-i|}{2} \right\rceil \right\}.$$

3 ID-Antiprisms and Their ID Numbers

As previously mentioned, our objectives are to characterize all ID-antiprisms, which are antiprism graphs that are ID-graphs, and to determine the ID number of any ID-antiprism. We now state our main result.

Theorem 3.1. *For $n \geq 3$, the graph AP_n is an ID-antiprism if and only if $n \geq 6$. Moreover, for $n \geq 6$,*

$$ID(AP_n) = \begin{cases} 3, & \text{if } n \geq 9 \text{ and } n \notin \{10, 12, 14, 16\}, \\ 4, & \text{if } n \in \{7, 8, 10, 12, 14, 16\}, \\ 6, & \text{if } n = 6. \end{cases}$$

The proof of the above theorem is split into several lemmas that are detailed in the rest of the paper. Specifically, Theorem 3.1 follows from Lemmas 3.1, 3.2, 3.4, and 3.6.

First, we establish that AP_3, AP_4, and AP_5 are not ID-graphs.

Lemma 3.1. *The antiprism graphs AP_3, AP_4, and AP_5 are not ID-graphs.*

Proof. For brevity, we prove the result for AP_3 only; the proofs for AP_4 and AP_5 can be carried out similarly. In light of Theorem 2.1 and since AP_3 is vertex-transitive, AP_3 cannot have an ID-coloring with exactly 1, 2, 4, or 5 red vertices. Thus, we are left to show that AP_3 cannot have an ID-coloring where exactly 3 vertices are red.

Consider a drawing of AP_3 where the vertices $0,2,4$ are in one row while $1,3,5$ are in another. Up to rotations and reflections, there are only 3 red-white colorings of AP_3 with exactly 3 red vertices: c_1 with red vertices $0,2,4$; c_2 with red vertices $0,1,2$; and c_3 with red vertices $0,2,3$. Under c_1, the multiset codes of 0 and 2 are both $\{0,1,1\}$. Under c_2, the multiset codes of 0 and 1 are both $\{0,1,1\}$. Under c_3, the multiset codes of 0 and 3 are both $\{0,1,2\}$. Thus, c_1, c_2, c_3 are all not ID-colorings of AP_3. □

In light of Theorem 2.1(a)–(b), it is clear that any ID-antiprism must have ID number at least 3. In Table 2, we present ID-colorings of antiprism graphs AP_n, where $n \in \{6,7,8,10,12,14,16\}$; thus, these antiprism graphs are ID-graphs. Moreover, using a similar approach as in Lemma 3.1, we can prove that ID$(AP_6) \geq 6$ while ID$(AP_n) \geq 4$ for $n \in \{7,8,10,12,14,16\}$. This leads to the following lemma.

Lemma 3.2. *If $n \in \{6,7,8,10,12,14,16\}$, then the antiprism graph AP_n is an ID-graph. Moreover, ID$(AP_6) = 6$ and ID$(AP_n) = 4$ if $n \in \{7,8,10,12,14,16\}$.*

Table 2: ID-colorings for antiprism graphs AP_n, where $n \in \{6,7,8,10,12,14,16\}$.

n	Red vertices in an ID-coloring of AP_n
6	$0,1,2,3,6,7$
7	$0,1,3,6$
8	$0,1,5,9$
10	$0,1,7,11$
12	$0,1,7,13$
14	$0,1,9,15$
16	$0,1,9,17$

In the following two subsections, we consider general antiprism graphs AP_n, where $n \geq 9$ and $n \notin \{10,12,14,16\}$.

3.1 ID number of AP_n, $n \geq 9$ and n is odd

Under this case, we show that AP_n has ID number equal to 3. In Figure 3, we verify that this is the case using a coloring of AP_9 and AP_{11} where the vertices 0, 1, and 7 are colored red.

As evident in Figure 3, the multiset codes in AP_{11} are closely related to the multiset codes in AP_9. We generalize this observation below.

Lemma 3.3. *Let $n \geq 9$ be an odd integer and let c be the red-white coloring of AP_n under which only the vertices 0, 1, and 7 are colored red. For any vertex v, let $M_n(v)$ be the multiset code of v as a vertex of AP_n under the coloring c. Then,*

$$M_{n+2}(i) = \begin{cases} M_n(i), & \text{if} \quad i \in \{0,1,\dots,n\}, \\ 1 + M_n(i-2), & \text{if} \quad i \in \{n+1, n+2, \dots, n+10\}, \\ M_n(i-4), & \text{if} \quad i \in \{n+11, n+12, \dots, 2n+3\}. \end{cases}$$

Proof. Fix $n \geq 9$. We proceed by cases.

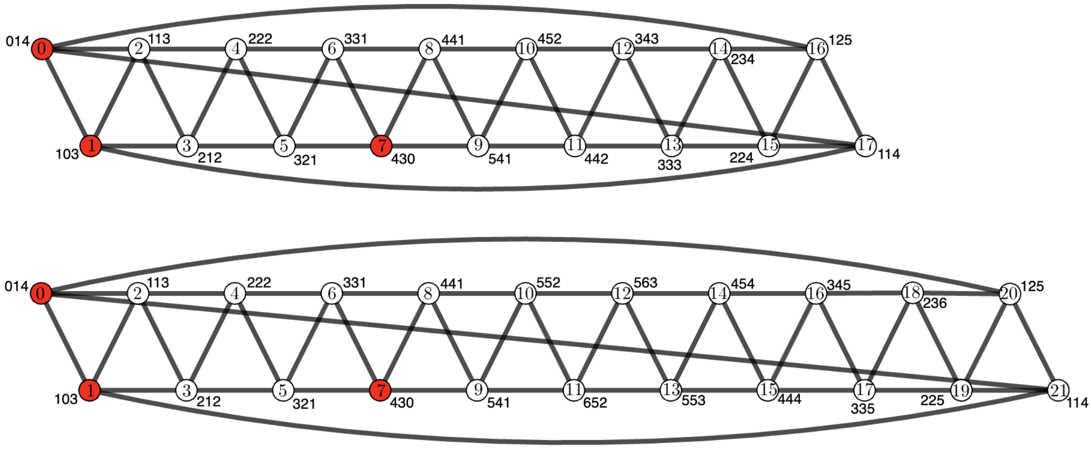

Figure 3: ID-colorings of the antiprism graphs AP_9 and AP_{11}; shown beside each vertex is its multiset code.

Case 1: Suppose $i \in \{0, 1, \ldots, 7\}$. Since $n \geq 9$, it follows immediately that, $M_{n+2}(i) = M_n(i) = \left\{ \left\lceil \frac{i}{2} \right\rceil, \left\lceil \frac{|i-1|}{2} \right\rceil, \left\lceil \frac{|i-7|}{2} \right\rceil \right\}$.

Case 2: Suppose $i \in \{8, 9, \ldots, n\}$. Let $k \in \{0, 1, 7\}$. Since $i \leq n$, then $i \leq n + k$, which implies $\frac{i-k}{2} \leq \frac{2n-(i-k)}{2} < \frac{2n+2-(i-k)}{2}$. Since k is arbitrary, we have $M_{n+2}(i) = M_n(i) = \left\{ \left\lceil \frac{i}{2} \right\rceil, \left\lceil \frac{i-1}{2} \right\rceil, \left\lceil \frac{i-7}{2} \right\rceil \right\}$, as desired.

Case 3: Suppose $i \in \{n+1, n+2, \ldots, n+10\}$. In this case, we want to prove that $M_{n+2}(i)$ is equal to the multiset $\{1 + x : x \in M_n(i-2)\}$. We introduce the following notation: For vertices v_1, v_2 that are both in AP_n and AP_{n+2}, let $d_n(v_1, v_2)$ and $d_{n+2}(v_1, v_2)$ be the distances between v_1 and v_2 in AP_n and AP_{n+2}, respectively.

Our desired result follows if we can show that $d_{n+2}(n+j, k) = d_n(n+j-2, k) + 1$ for all $k \in \{0, 1, 7\}$ and $j \in \{1, 2, \ldots, 10\}$. Fix such k and j arbitrarily; note that $n + j > n + j - 2 > k$. Using the formula in Observation 2.2, we have,

$$d_{n+2}(n+j, k) = \min \left\{ \left\lceil \frac{n+j-k}{2} \right\rceil, \left\lceil \frac{n-j+k+4}{2} \right\rceil \right\},$$

while,

$$d_n(n+j-2, k) = \min \left\{ \left\lceil \frac{n+j-k-2}{2} \right\rceil, \left\lceil \frac{n-j+k+2}{2} \right\rceil \right\}.$$

If $1 \leq j \leq k+2$, then $n+j-k \leq n-j+k+4$ and equivalently, $n+j-k-2 \leq n-j+k+2$. Thus,

$$d_{n+2}(n+j, k) = \left\lceil \frac{n+j-k}{2} \right\rceil = \left\lceil \frac{n+j-k-2}{2} \right\rceil + 1 = d_n(n+j-2, k) + 1,$$

as desired. On the other hand, if $k + 3 \leq j \leq 10$, then $n + j - k > n - j + k + 4$ and equivalently, $n + j - k - 2 > n - j + k + 2$. The desired conclusion also follows in this case since,

$$d_{n+2}(n+j, k) = \left\lceil \frac{n-j+k+4}{2} \right\rceil = \left\lceil \frac{n-j+k+2}{2} \right\rceil + 1 = d_n(n+j-2, k) + 1.$$

Case 4: Suppose $i \in \{n+11, n+12, \ldots, 2n+3\}$. Similar to the previous case, it is enough to show that $d_{n+2}(n+j,k) = d_n(n+j-4,k)$ for all $k \in \{0,1,7\}$ and $j \in \{11, 12, \ldots, n+3\}$. Fix such k and j arbitrarily; note that $n+j > n+j-4 > k$. Then,

$$d_{n+2}(n+j,k) = \min\left\{\left\lceil \frac{n+j-k}{2} \right\rceil, \left\lceil \frac{n-j+k+4}{2} \right\rceil\right\},$$

while,

$$d_n(n+j-4,k) = \min\left\{\left\lceil \frac{n+j-k-4}{2} \right\rceil, \left\lceil \frac{n-j+k+4}{2} \right\rceil\right\}.$$

Since $j \geq 11 \geq k+4$, it follows that $n+j-k > n+j-k-4 \geq n-j+k+4$. Thus, $d_{n+2}(n+j,k) = d_n(n+j-4,k) = \left\lceil \frac{n-j+k+4}{2} \right\rceil$, as desired.

□

The next lemma immediately implies that the ID number of AP_n, in this case, is equal to 3.

Lemma 3.4. *Let $n \geq 9$ be an odd integer. Then, the red-white coloring of AP_n under which only the vertices 0, 1, and 7 are colored red is an ID-coloring of AP_n.*

Proof. The cases where $n = 9$ and $n = 11$ are covered in Figure 3. We use $n = 11$ as the base case. Suppose the desired result is true for some odd integer $n \geq 11$. We now consider AP_{n+2}.

Let $i, i' \in V(AP_{n+2})$ such that $i < i'$. Recall the notations $M_n(\cdot)$ and $d_n(\cdot)$ introduced in Lemma 3.3 and its proof.

Case 1: Suppose one of the sets $\{0, \ldots, n\}$, $\{n+1, \ldots, n+10\}$, $\{n+11, \ldots, 2n+3\}$ contain both i and i'. Then, by Lemma 3.3 and the inductive hypothesis, we must have $M_{n+2}(i) \neq M_{n+2}(i')$.

Case 2: Suppose $i \in \{0, \ldots, n\}$ and $i' \in \{n+1, \ldots, n+10\}$. Recall from the proof of Lemma 3.3 that $M_{n+2}(i) = \left\{\left\lceil \frac{i}{2} \right\rceil, \left\lceil \frac{|i-1|}{2} \right\rceil, \left\lceil \frac{|i-7|}{2} \right\rceil\right\}$. Thus, if $0 \leq i \leq 7$, then $\min M_{n+2}(i) \leq 2$; on the other hand, if $8 \leq i \leq n$, then $\min M_{n+2}(i) = \left\lceil \frac{i-7}{2} \right\rceil$. In either case, we have $\min M_{n+2}(i) \leq \frac{n-7}{2}$. Thus, the desired result follows if we can show that $\min M_{n+2}(i') > \frac{n-7}{2}$.

It is enough to show that $d_{n+2}(n+j,k) > \frac{n-7}{2}$ for all $k \in \{0,1,7\}$ and $j \in \{1, 2, \ldots, 10\}$. Fix such k and j arbitrarily. From the proof of Lemma 3.3, we have,

$$d_{n+2}(n+j,k) = \min\left\{\left\lceil \frac{n+j-k}{2} \right\rceil, \left\lceil \frac{n-j+k+4}{2} \right\rceil\right\}.$$

Since $1 \leq j \leq 10$ and $0 \leq k \leq 7$, we have $\frac{n+j-k}{2} \geq \frac{n+1-7}{2} = \frac{n-6}{2}$ and $\frac{n-j+k+4}{2} \geq \frac{n-10+4}{2} = \frac{n-6}{2}$. Thus, since n is odd, we have $d_{n+2}(n+j,k) \geq \left\lceil \frac{n-6}{2} \right\rceil = \frac{n-5}{2}$, as desired.

Case 3: Suppose $i \in \{n+1, \ldots, n+10\}$ and $i' \in \{n+11, n+12, \ldots, 2n+3\}$. Let $i' = n+j$, where $j \in \{11, 12, \ldots, n+3\}$. For $k \in \{0, 1, 7\}$, Case 4 in the proof of Lemma 3.3 implies that $d_{n+2}(n+j, k) = \left\lceil \frac{n-j+k+4}{2} \right\rceil$. Consequently, $\min M_{n+2}(n+j) = \left\lceil \frac{n-j+4}{2} \right\rceil \leq \left\lceil \frac{n-11+4}{2} \right\rceil = \frac{n-7}{2}$. The rest of the proof for this case proceeds similarly as in Case 2.

Case 4: Suppose $i \in \{0, \ldots, n\}$ and $i' \in \{n+11, n+12, \ldots, 2n+3\}$. Note that $i' - 4 > i$. Thus, by Lemma 3.3 and the inductive hypothesis, we have $M_{n+2}(i) = M_n(i) \neq M_n(i'-4) = M_{n+2}(i')$.

□

3.2 ID number of AP_n, $n \geq 18$ and n is even

As in the previous subsection, our goal is to prove that AP_n, $n \geq 18$ and n is even, has ID number equal to 3. For each such n, define c_n to be the red-white coloring of AP_n where only the vertices $0, 8$, and $n+2$ are colored red. The following formulas can be derived by applying Observation 2.2.

Observation 3.2. *Let i be a vertex of AP_n, where $n \geq 18$ and n is even. Then, the distances of i to the vertices $0, 8$, and $n+2$ are given by the following:*

$$d(i,0) = \begin{cases} \left\lceil \dfrac{i}{2} \right\rceil, & i \in \{0, 1, \ldots, n\}, \\[2ex] \left\lceil \dfrac{2n-i}{2} \right\rceil, & i \in \{n+1, n+2, \ldots, 2n-1\}, \end{cases}$$

$$d(i,8) = \begin{cases} \left\lceil \dfrac{8-i}{2} \right\rceil, & i \in \{0, 1, \ldots, 8\}, \\[2ex] \left\lceil \dfrac{i-8}{2} \right\rceil, & i \in \{9, 10, \ldots, n+8\}, \\[2ex] \left\lceil \dfrac{2n-i+8}{2} \right\rceil, & i \in \{n+9, n+10, \ldots, 2n-1\}, \end{cases}$$

$$d(i,n+2) = \begin{cases} \left\lceil \dfrac{n-2+i}{2} \right\rceil, & i \in \{0, 1\}, \\[2ex] \left\lceil \dfrac{n+2-i}{2} \right\rceil, & i \in \{2, 3, \ldots, n+2\}, \\[2ex] \left\lceil \dfrac{i-n-2}{2} \right\rceil, & i \in \{n+3, n+4, \ldots, 2n-1\}. \end{cases}$$

In order to establish that c_n is an ID-coloring, our argument uses comparisons involving the $\text{sum}(\cdot)$ and $\text{sum}_2(\cdot)$ of the vertices. Given a sequence (v_k) of vertices in AP_n, we define

$\mathrm{sum}(v_k) := (\mathrm{sum}(v_k))$ and $\mathrm{sum}_2(v_k) := (\mathrm{sum}_2(v_k))$. Lastly, given a sequence (a_j), where each $a_j \in \mathbb{Z}$, and $b \in \mathbb{Z}$, we denote by $b + (a_j)$ the sequence $(b + a_j)$.

In the following lemma, we completely determine $\mathrm{sum}(\cdot)$ and $\mathrm{sum}_2(\cdot)$ for each vertex in AP_n.

Lemma 3.5. *Let c_n, where $n \geq 18$ and n is even, be the red-white coloring of AP_n where the only red vertices are $0, 8$, and $n+2$. Then,*

$$\begin{aligned}
\mathrm{sum}(0,\ldots,8) &= \frac{n}{2} + (3,5,4,5,3,4,2,3,1),\\
\mathrm{sum}(9,\ldots,n) &= \frac{n}{2} + \left(3,2,4,3,\ldots,\frac{n}{2}-2,\frac{n}{2}-3\right),\\
\mathrm{sum}(n+1,\ldots,n+7) &= \frac{n}{2} + \left(\frac{n}{2}-2,\frac{n}{2}-4,\frac{n}{2}-2,\frac{n}{2}-3,\frac{n}{2}-1,\frac{n}{2}-2,\frac{n}{2}\right),\\
\mathrm{sum}(n+8,\ldots,2n-1) &= \frac{n}{2} + \left(\frac{n}{2}-1,\frac{n}{2},\frac{n}{2}-2,\frac{n}{2}-1,\ldots,4,5\right).
\end{aligned}$$

Moreover, if $n \equiv 0 \pmod 4$, then,

$$\begin{aligned}
\mathrm{sum}_2(0,1,\ldots,8) &= (4,5,4,5,\ldots,4,5,4),\\
\mathrm{sum}_2\left(9,10,\ldots,\frac{n}{2}\right) &= \left(6,6,8,8,\ldots,\frac{n}{2}-4,\frac{n}{2}-4\right),\\
\mathrm{sum}_2\left(\frac{n}{2}+1,\ldots,n+2\right) &= \frac{n}{2} + (-2,-3,-2,-3,\ldots,-2,-3),\\
\mathrm{sum}_2\left(n+3,\ldots,\frac{3n}{2}+5\right) &= \frac{n}{2} + (-1,-1,0,-1,0,\ldots,-1,0),\\
\mathrm{sum}_2\left(\frac{3n}{2}+6,\ldots,2n-1\right) &= \left(\frac{n}{2}-2,\frac{n}{2}-2,\ldots,8,8,6,6\right),
\end{aligned}$$

while if $n \equiv 2 \pmod 4$, then,

$$\begin{aligned}
\mathrm{sum}_2(0,1,\ldots,8) &= (4,5,4,5,\ldots,4,5,4),\\
\mathrm{sum}_2\left(9,10,\ldots,\frac{n}{2}+1\right) &= \left(6,6,8,8,\ldots,\frac{n}{2}-3,\frac{n}{2}-3\right),\\
\mathrm{sum}_2\left(\frac{n}{2}+2,\ldots,n+2\right) &= \frac{n}{2} + (-2,-3,-2,-3,\ldots,-2,-3),\\
\mathrm{sum}_2\left(n+3,\ldots,\frac{3n}{2}+4\right) &= \frac{n}{2} + (-1,-1,0,-1,0,\ldots,-1,0),\\
\mathrm{sum}_2\left(\frac{3n}{2}+5,\ldots,2n-1\right) &= \left(\frac{n}{2}-1,\frac{n}{2}-1,\ldots,8,8,6,6\right).
\end{aligned}$$

Sketch of proof. The given formulas follow by applying Observation 3.2. To illustrate, suppose $n \equiv 0 \pmod 4$ and let i be a vertex of AP_n such that $i \in \{9,10,\ldots,n\}$. Then,

$$M(i) = \left\{ \left\lceil \frac{i}{2} \right\rceil, \left\lceil \frac{i-8}{2} \right\rceil, \left\lceil \frac{n+2-i}{2} \right\rceil \right\}.$$

If i is even, then $\mathrm{sum}(i) = \frac{i}{2} + \frac{i-8}{2} + \frac{n+2-i}{2} = \frac{n}{2} + \frac{i-6}{2}$; while if i is odd, then $\mathrm{sum}(i) = \frac{i+1}{2} + \frac{i-7}{2} + \frac{n+3-i}{2} = \frac{n}{2} + \frac{i-3}{2}$. From these, we have $\mathrm{sum}(9,\ldots,n) = \frac{n}{2} + \left(3,2,4,3,\ldots,\frac{n}{2}-2,\frac{n}{2}-3\right)$.

Now, suppose further that $i \le n/2$; that is, $i \in \{9, 10, \ldots, n/2\}$. If i is even, then
$$\text{sum}_2(i) = \min\left\{\frac{i}{2} + \frac{i-8}{2}, \frac{i}{2} + \frac{n+2-i}{2}, \frac{i-8}{2} + \frac{n+2-i}{2}\right\} = \min\{i-4, \frac{n+2}{2}, \frac{n-6}{2}\} = i-4$$
since $i \le n/2$. Similarly, if i is odd, then $\text{sum}_2(i) = \min\left\{i-3, \frac{n+4}{2}, \frac{n-4}{2}\right\} = i-3$. It now follows that $\text{sum}_2\left(9, 10, \ldots, \frac{n}{2}\right) = \left(6, 6, 8, 8, \ldots, \frac{n}{2} - 4, \frac{n}{2} - 4\right)$, as desired.

The other formulas can be proven using similar approaches. $\square$

In Table 3, we illustrate the red-white coloring c_{18} of AP_{18}, including the multiset code, the sum$(\cdot)$, and the $\text{sum}_2(\cdot)$ of each vertex. The results of Lemma 3.5 can also be verified from Table 3.

Table 3: $M(v)$, sum(v), $\text{sum}_2(v)$ for each vertex v of AP_{18} under the coloring c_{18}, in which 0, 8, and 20 are colored red.

v	$M(v)$	sum(v)	$\text{sum}_2(v)$	v	$M(v)$	sum(v)	$\text{sum}_2(v)$
0	$\{0,4,8\}$	12	4	18	$\{1,5,9\}$	15	6
1	$\{1,4,9\}$	14	5	19	$\{1,6,9\}$	16	7
2	$\{1,3,9\}$	13	4	20	$\{0,6,8\}$	14	6
3	$\{2,3,9\}$	14	5	21	$\{1,7,8\}$	16	8
4	$\{2,2,8\}$	12	4	22	$\{1,7,7\}$	15	8
5	$\{2,3,8\}$	13	5	23	$\{2,7,8\}$	17	9
6	$\{1,3,7\}$	11	4	24	$\{2,6,8\}$	16	8
7	$\{1,4,7\}$	12	5	25	$\{3,6,9\}$	18	9
8	$\{0,4,6\}$	10	4	26	$\{3,5,9\}$	17	8
9	$\{1,5,6\}$	12	6	27	$\{4,5,9\}$	18	9
10	$\{1,5,5\}$	11	6	28	$\{4,4,8\}$	16	8
11	$\{2,5,6\}$	13	7	29	$\{4,5,8\}$	17	9
12	$\{2,4,6\}$	12	6	30	$\{3,5,7\}$	15	8
13	$\{3,4,7\}$	14	7	31	$\{3,6,7\}$	16	9
14	$\{3,3,7\}$	13	6	32	$\{2,6,6\}$	14	8
15	$\{3,4,8\}$	15	7	33	$\{2,6,7\}$	15	8
16	$\{2,4,8\}$	14	6	34	$\{1,5,7\}$	13	6
17	$\{2,5,9\}$	16	7	35	$\{1,5,8\}$	14	6

We are now ready to prove that c_n is indeed an ID-coloring of AP_n, for each even $n \ge 18$. Recall that if two vertices have different sum$(\cdot)$ or different $\text{sum}_2(\cdot)$, then their multiset codes are also different.

Lemma 3.6. *Let c_n, where $n \ge 18$ and n is even, be the red-white coloring of AP_n where the only red vertices are $0, 8$, and $n+2$. Then, c_n is an ID-coloring of AP_n.*

Proof. Suppose $n \ge 20$, $n \equiv 0 \pmod 4$, and that AP_n has been colored using the red-white coloring c_n. Let $i, j \in V(AP_n)$ such that $i < j$. We will prove that $M(i) \ne M(j)$ under all possible cases. Our arguments will use the formulas in Lemma 3.5.

Case 1: Suppose $i \in \{0, \ldots, 8\}$.

Case 1.1: Suppose $j \in \{0, \ldots, 8\}$. If i and j are of different parity, then $\text{sum}_2(i) \neq \text{sum}_2(j)$. If i and j are both even, then $\text{sum}(i) \neq \text{sum}(j)$ except when $(i, j) = (0, 4)$; and $M(0) \neq M(4)$ since 0 and 4 have different colors. If i and j are both odd, then $\text{sum}(i) \neq \text{sum}(j)$ except when $(i, j) = (1, 3)$; and $M(1) = (1, 4, n/2) \neq (2, 3, n/2) = M(3)$.

Case 1.2: Suppose $j \in \{9, \ldots, 2n-1\}$. Since $\text{sum}_2(i) \leq 5$ while $\text{sum}_2(j) \geq 6$, it follows that $\text{sum}_2(i) \neq \text{sum}_2(j)$.

Case 2: Suppose $i \in \left\{9, \ldots, \frac{n}{2}\right\}$.

Case 2.1: Suppose $j \in \left\{9, \ldots, \frac{n}{2}\right\}$. In this case, notice that $\text{sum}_2(i) = \text{sum}_2(j)$ if and only if i is odd and $j = i+1$. However, for odd i, it is clear that $\text{sum}(i) \neq \text{sum}(i+1)$.

Case 2.2: Suppose $j \in \left\{\frac{n}{2}+1, \ldots, \frac{3n}{2}+5\right\}$. Since $\text{sum}_2(i) \leq \frac{n}{2} - 4$ while $\text{sum}_2(j) \geq \frac{n}{2} - 3$, then $\text{sum}_2(i) \neq \text{sum}_2(j)$.

Case 2.3: Suppose $j \in \left\{\frac{3n}{2}+6, \ldots, 2n-1\right\}$. If $j = \frac{3n}{2} + 6$ or $\frac{3n}{2} + 7$, then $\text{sum}_2(j) = \frac{n}{2} - 2 > \frac{n}{2} - 4 \geq \text{sum}_2(i)$. Thus, we are left to consider $j \in \left\{\frac{3n}{2}+8, \ldots, 2n-1\right\}$. Note that $\text{sum}_2\left(9, 10, \ldots, \frac{n}{2}\right)$ $= \left(6, 6, 8, 8, \ldots, \frac{n}{2}-4, \frac{n}{2}-4\right) = \text{sum}_2\left(2n-1, 2n-2, \ldots, \frac{3n}{2}+8\right)$. Moreover, $\text{sum}\left(9, 10, \ldots, \frac{n}{2}\right) = \frac{n}{2} + \left(3, 2, 4, 3, \ldots, \frac{n}{4}-2, \frac{n}{4}-3\right)$ while $\text{sum}\left(2n-1, 2n-2, \ldots, \frac{3n}{2}+8\right) = \frac{n}{2} + \left(5, 4, 6, 5, \ldots, \frac{n}{4}, \frac{n}{4}-1\right)$. Thus, $\text{sum}(i) \neq \text{sum}(j)$ if $\text{sum}_2(i) = \text{sum}_2(j)$.

Case 3: Suppose $i \in \left\{\frac{n}{2}+1, \ldots, n+2\right\}$.

Case 3.1: Suppose $j \in \left\{\frac{n}{2}+1, \ldots, n\right\}$. Notice that $\text{sum}_2(i) = \text{sum}_2(j)$ if and only if i and j have the same parity. Moreover, it can be seen that $\text{sum}(9, 11, \ldots, n-1)$ and $\text{sum}(10, 12, \ldots, n)$ are increasing sequences. Thus, $\text{sum}(i) \neq \text{sum}(j)$ whenever $\text{sum}_2(i) = \text{sum}_2(j)$.

Case 3.2: Suppose $j \in \{n+1, n+2\}$. Again, $\text{sum}_2(i) = \text{sum}_2(j)$ if and only if i and j have the same parity. This leaves us to consider the cases where $(i, j) \in \{(n-1, n+1), (n-2, n+2)\}$. Since $\min M(n-1) = 2 \neq 1 = \min M(n+1)$, then $M(n-1) \neq M(n+1)$. Meanwhile, $M(n) \neq M(n+2)$ and $M(n-2) \neq M(n+2)$ since $n-2$ and n are colored white while $n+2$ is colored red.

Case 3.3: Suppose $j \in \left\{n+3, \ldots, \frac{3n}{2}+5\right\}$. Since $\text{sum}_2(i) \leq \frac{n}{2} - 2$ while $\text{sum}_2(j) \geq \frac{n}{2} - 1$, then $\text{sum}_2(i) \neq \text{sum}_2(j)$.

Case 3.4: Suppose $j \in \left\{\frac{3n}{2}+6, \ldots, 2n-1\right\}$. Then $\text{sum}_2(i) = \text{sum}_2(j)$ if and only if i is odd and $j \in \left\{\frac{3n}{2}+6, \frac{3n}{2}+7\right\}$. Thus, we are left to consider these values of i and j. Note that $\text{sum}\left(\frac{3n}{2}+6\right) = \frac{3n}{4}$ and $\text{sum}\left(\frac{3n}{2}+7\right) = \frac{3n}{4} + 1$.

Looking for values of i with the same $\mathrm{sum}(\cdot)$, we find that the remaining cases to consider are $(i,j) \in \left\{ \left(\frac{n}{2}+3, \frac{3n}{2}+6\right), \left(\frac{n}{2}+5, \frac{3n}{2}+7\right) \right\}$. To complete the proof for this case, we have,

$$M\left(\frac{n}{2}+3\right) = \left\{\frac{n}{4}+2, \frac{n}{4}-2, \frac{n}{4}\right\} \neq \left\{\frac{n}{4}-3, \frac{n}{4}+1, \frac{n}{4}+2\right\} = M\left(\frac{3n}{2}+6\right),$$

and

$$M\left(\frac{n}{2}+5\right) = \left\{\frac{n}{4}+3, \frac{n}{4}-1, \frac{n}{4}-1\right\} \neq \left\{\frac{n}{4}-3, \frac{n}{4}+1, \frac{n}{4}+3\right\} = M\left(\frac{3n}{2}+7\right).$$

Case 4: Suppose $i \in \left\{n+3, \dots, \frac{3n}{2}+5\right\}$. If $j \in \left\{n+3, \dots, \frac{3n}{2}+5\right\}$ also, then the proof proceeds similarly as the one for Case 3.1 except when $i, j \in \{n+3, n+6, n+10\}$ but, these cases can be addressed easily. If $j \in \left\{\frac{3n}{2}+6, \dots, 2n-1\right\}$, then clearly, $\mathrm{sum}_2(i) > \mathrm{sum}_2(j)$.

Case 5: Suppose $i, j \in \left\{\frac{3n}{2}+6, \dots, 2n-1\right\}$. The proof of this case proceeds similarly as the one for Case 2.1.

Finally, the proof for the case where $n \equiv 2 \pmod 4$ follows a similar approach. $\square$

4 Conclusions

The notion of ID-colorings (equivalently, multiset resolving sets) can be used to solve the problem of uniquely identifying the vertices of a graph. In this paper, we have considered antiprism graphs, which are of particular interest because they are known to be vertex-transitive (thus, possessing several automorphisms). Our main result (Theorem 3.1) is a characterization of all antiprism graphs that possess at least one ID-coloring. More specifically, we have proved that an n-antiprism graph is an ID-graph if and only if $n \geq 6$. Moreover, we have completely determined the identification numbers of all ID-antiprisms. We have found that almost all ID-antiprisms have ID number equal to 3; the exceptions are $AP_7, AP_8, AP_{10}, AP_{12}, AP_{14}, AP_{16}$ that have ID number 4 and AP_6 that has ID number 6.

Acknowledgement The author thanks the Department of Mathematics, the OAVP for Research, Creative Work, and Innovation (RCWI), and the University Research Council of the Ateneo de Manila University for supporting this work through his RCW Faculty Grant RCW 08 2023. The author also thanks the OAVPRCWI, the School of Science and Engineering, and the Department of Mathematics BCA Fund for supporting the author's attendance to and presentation at The International Conference on Mathematical Sciences and Technology 2024 (MathTech 2024). The author also expresses his sincere gratitude to the expert and generous reviewers and editors whose comments and guidance have improved this paper. Finally, the author expresses his appreciation to all the committee members of MathTech 2024 for organizing a wonderful conference.

Conflicts of Interest The author declares no conflict of interest.

References

[1] G. Chartrand, Y. Kono & P. Zhang (2021). Distance vertex identification in graphs. *Journal of Interconnection Networks*, *21*(1), Article ID: 2150005. https://doi.org/10.1142/S0219265921500055.

[2] P. R. Cromwell (1997). *Polyhedra*. Cambridge University Press, Cambridge, United Kingdom.

[3] R. El-Shanawany & A. El-Mesady (2018). On orthogonal labelling for the orthogonal covering of the circulant graphs. *Malaysian Journal of Mathematical Sciences*, *12*(2), 161–173.

[4] A. Hakanen & I. G. Yero (2024). Complexity and equivalency of multiset dimension and ID-colorings. *Fundamenta Informaticae*, *191*(3-4), 315–330. https://doi.org/10.3233/FI-242185.

[5] Y. Kono & P. Zhang (2022). Vertex identification in grids and prisms. *Journal of Interconnection Networks*, *22*(2), Article ID: 2150019. https://doi.org/10.1142/S0219265921500195.

[6] R. M. Marcelo, M. A. C. Tolentino, A. D. Garciano, M. J. P. Ruiz & J. C. Buot (2025). On the vertex identification spectra of grids. *Journal of Interconnection Networks*, *25*(1), Article ID: 2450002. https://doi.org/10.1142/S0219265924500026.

[7] R. C. Read & R. J. Wilson (1998). *An Atlas of Graphs*. Oxford University Press, Oxford, United Kingdom.

[8] V. Saenpholphat (2009). On multiset dimension in graphs. *Academic Srinakharinwirot University*, *1*, 193–202.

[9] H. Saleem, M. Husin, S. Ali & M. Hameed (2025). On examining metric dimension through edge contraction in certain families of graphs. *Malaysian Journal of Mathematical Sciences*, *19*(2), 613–635. https://doi.org/10.47836/mjms.19.2.13.

[10] R. Simanjuntak, P. Siagian & T. Vetrik. The multiset dimension of graphs. arXiv: Combinatorics 2017. https://doi.org/10.48550/arXiv.1711.00225.